Results from a Detailed Calculation of the Sensitivity of a Cryogenic Current Comparator

Murray D. Early and Marcos A. van Dam

***Abstract*— A detailed calculation of the sensitivity of a cryogenic current comparator (CCC) has been completed. The model accounts for the current distributions on the closed external shield and the overlapped shield around the windings. Additional refinements to the model lead to excellent agreement (typically better than 5%) with sensitivity measurements for a wide range of CCC geometries. We demonstrate that generally the best strategy for significantly improving the overall gain of a CCC is to increase the number of windings in the toroid at the expense of a reduction in the sensitivity.**

***Index Terms*—Cryogenic current comparators, cryogenic electronics, current distributions, inductance calculations, superconducting shielding.**

I. Introduction

There is widespread and successful use of cryogenic current comparators (CCC's) for the measurement of the resistance generated by the quantum Hall effect. The basic concept of a CCC would make it also appear as an ideal candidate for the precise measurement of the small currents generated by single electron tunneling devices [1]. However in practice the lack of sensitivity has proved to be a limitation. Progress in resolving this issue is in part hampered by the absence of an accurate method for the calculation of the CCC sensitivity.

We have previously published a simplified sensitivity calculation for an externally shielded CCC [2], with a claimed accuracy of around 20%. In order to make use of an existing fast Fourier transform (FFT) method for dealing with the current distribution of the external shield, this calculation treated the CCC toroid and the pick-up coil as single filaments (we refer to this as the FFT model). In addition a number of approximations were made in this model that were likely to be significant at the claimed level of accuracy. We have since developed a detailed model of the entire CCC structure that accounts for nearly all of the effects that had previously been neglected (we refer to this as the strip model). The mathematical details of this reasonably complex model will be described in [3]. Here we report some results from the strip model that can be compared with measured values of the sensitivity for our own CCC, as well as for a range of CCC geometries that have been reported in the literature [1], [4].

Manuscript received July 2, 1998.

The authors are with the Measurement Standards Laboratory of New Zealand, Industrial Research Limited, Lower Hutt, New Zealand.

Publisher Item Identifier S 0018-9456(99)03197-6.

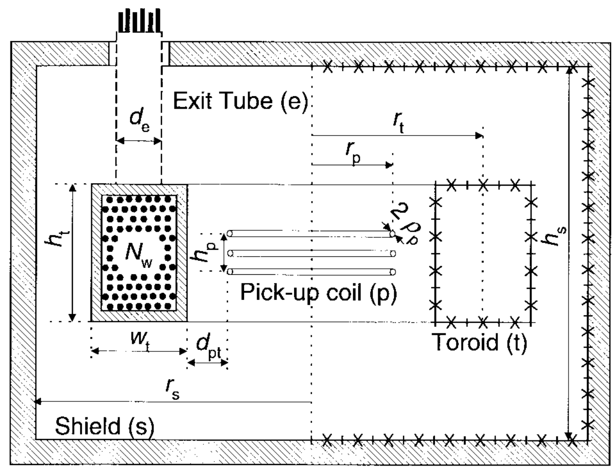

Fig. 1. Cross section of the CCC described by the strip model. The right half of the cross section shows how the surfaces of the external shield and toroid are divided into strips with a filament (X) at their center.

II. Model Parameters

The cross section of the cylindrically symmetric CCC described by the model is shown in Fig. 1. The N_w ratio windings of the CCC are surrounded by a superconducting toroid of rectangular cross section (height h_t and width w_t). The windings enter and exit the toroid through a cylindrical exit tube of diameter d_e. A multi-turn superconducting pick-up coil is connected in series with the input inductance $(L_{\rm SQ})$ of a SQUID. The N_p helical turns of the pick-up coil are treated as a series of axial loops (of axial radius r_p and bore radius ρ_p) evenly spaced over height h_p. A superconducting external shield of internal radius r_s and height h_s encloses the CCC.

To account for the current distribution on the toroid and the external shield, the surfaces are divided into a series of cylindrical strips (see Fig. 1). For well-separated strips the current can be considered to be concentrated at a filament at the center of each strip. For nearby strips the effective separation of the current filaments is given by the geometric mean distance (GMD) of the strips [5].

III. Sensitivity Calculation

To calculate the sensitivity we must simultaneously solve for the current of each of the circular strips (or filaments) that constitute the shield, toroid, and pick-up coil. This is achieved by setting up a matrix of mutual inductances M and solving $M \cdot \mathbf{i} = \varphi$ for the vector of currents $\mathbf{i}$. The flux vector φ is

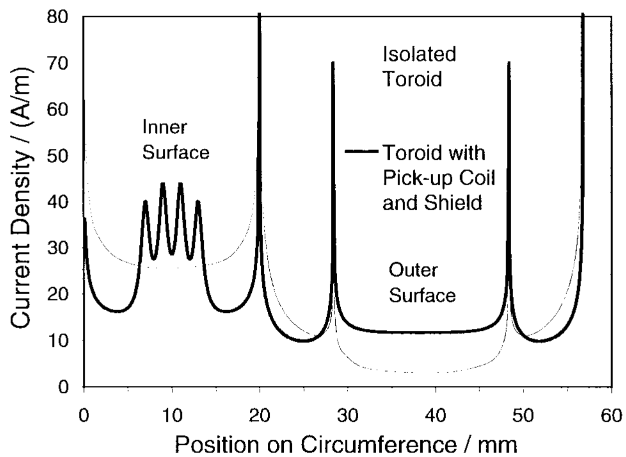

Fig. 2. Current distribution around the circumference of the toroid of the CCC in [2]. Also shown is the distribution for an isolated toroid (in the absence of the shield and the pick-up coil). In each case the total current is 1 A.

zero for all closed superconducting loops (such as those of the external shield, and the loop containing the pick-up coil and the SQUID) and constant for all of the toroid loops. The elements of M are given by a standard expression for the mutual inductance of two concentric loops [3].

The sensitivity (the ratio of the pick-up coil current i_p to the toroid current i_t) is then given by

$$\left(\frac{i_p}{i_t}\right) = \frac{\mathbf{i}_{\text{pickup-coil}}}{\sum_{\text{toroid}} \mathbf{i}}. \tag{1}$$

In the FFT model, the optimum value of N_p could be calculated analytically. However, this is not possible in the strip model as N_p must be known in advance to calculate the toroid current distribution. As a result N_p must be independently varied to find the best value.

IV. Calculated Current Distributions

The calculated toroid current distribution for the CCC described in [2] is shown in Fig. 2 with, and without, the external shield and pick-up coil. The parameters used to generate this data are given in Table I. For the isolated toroid the current distribution is similar to that calculated by Symm [6]. It is evident that the presence of the external shield and pick-up coil causes the toroid current to be drawn toward these surfaces to cancel the flux generated by the current in these elements. This demonstrates how the shield reduces sensitivity and emphasizes the need for the pick-up coil to be closely coupled to the toroid.

The strong variation in the current density is not consistent with the assumption that the current is uniform across the strips. However, this has a negligible effect if a large number of strips are used to model the toroid. In practice we find that using more than about 200 strips to model the toroid ensures that the calculated sensitivity is within about 0.2% of the limiting value. It is also important to ensure that the width of the strips is much smaller than d_{pt}, the gap between the pick-up coil and the toroid, so that the sensitivity does not depend on the position of the strips.

TABLE I
Parameters and Calculated Sensitivity Results for the CCC's Built by Early and Jones [2] and Hartland [4]. The Parameters That Have Been Estimated for the Hartland CCC Are Shown in Bold

Parameter	Early and Jones [2]	Hartland [4]	
		Constructed	Optimised
r_t (mm)	20.3	80.0	85.0
h_t (mm)	20.0	**30.2**	31.0
w_t (mm)	8.4	**23.4**	22.8
r_p (mm)	15.5	67.5	72.5
h_p (mm)	6.0	**10.0**	24.8
ρ_p (mm)	0.5	0.0625	0.9
N_p	3.8	3	4
r_s (mm)	29.9	**120.0**	fixed
h_s (mm)	49.2	**120.0**	240.0
L_{SQ} (μH)	0.808	2.2	fixed
L_{leads} (μH)	0.056	**-0.015**	fixed
Sensitivity:			
Measured	0.043	0.0858	-
Calculated	0.0448	0.0858	0.1116

TABLE II
Parameters and Calculated Sensitivity Results for the CCC's built by Pesel *et al.* [4] Where $L_{SQ} = 2\ \mu$H and $\rho_p = 0.075$ mm. The Following Additional Information on the Construction of These Devices Has Been Obtained from E. Pesel: The SQUID Sensitivity Is 0.15 μA/φ_o (Manufacturer's Report), $r_s = 30$ mm, $h_s = 40$ mm, $h_p = 8$ mm, and $d_e = 4.4$ mm

CCC Label	r_t (mm)	h_t (mm)	w_t (mm)	r_p (mm)	N_p	Measured Sensitivity	Calculated Sensitivity
J1	15.5	20.0	13.5	8.48	8	0.0177	0.0187
D7	13.3	12.5	8.0	8.48	8	0.0231	0.0228
D9a	13.5	12.5	8.0	8.43	8	0.0228	0.0219
D9b	13.5	12.5	8.0	8.43	12	0.0238	0.0240
D11a	20.0	12.5	8.0	15.83	12	0.0405	0.0411
D11b	20.0	12.5	8.0	15.83	8	0.0417	0.0414

Fig. 3 compares the results for four CCC configurations using the two models. The geometries used are the same as those in Fig. 4 of [2][1] with the exception that N_p is now a constant for each curve, having been set to the value associated with the peak of the curves in [2]. The FFT model treats the toroid as a single filament, which is appropriate for a circular cross section (of radius ρ_t) with a uniform current. For the strip model we take a square cross section of the same area ($h_t^2 = w_t^2 = \pi\rho_t^2$). This difference in shape is not particularly significant [6]. In addition, the FFT model assumes that the turns of the pick-up coil are coincident while in this case the strip model treats the turns as adjacent with $h_p = 2\rho_p(N_p - 1)$.

The two notable features of Fig. 3 are that the strip model results have an increase in the peak sensitivity (by about 30% for the highest curve) and a much sharper decline in sensitivity as the outer surface of the toroid approaches the external

[1] Some of the parameters given in the caption of Fig. 4 in [2] are incorrect. The corrected parameters are given in Fig. 3 of this article although N_p is allowed to vary in [2].

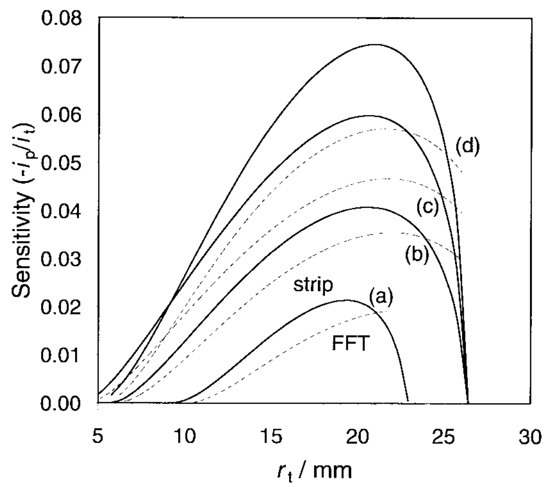

Fig. 3. Dependence of sensitivity $(-i_p/i_t)$ on toroid radius r_t for a range of parameters, calculated using both the FFT model (dotted lines) and the strip model (solid lines). The appropriate curve label is drawn near the intersection of the two model curves. For curves (a) $\rho_t = 8$ mm, $\rho_p = 0.065$ mm, $d_{pt} = 2$ mm, and $N_p = 3.3$; (b) $\rho_t = 4$ mm, $\rho_p = 0.065$ mm, $d_{pt} = 2$ mm, and $N_p = 2.8$; (c) $\rho_t = 4$ mm, $\rho_p = 0.065$ mm, $d_{pt} = 0.25$ mm, and $N_p = 2.7$; (d) $\rho_t = 4$ mm, $\rho_p = 1$ mm, $d_{pt} = 0.25$ mm, and $N_p = 4.1$.

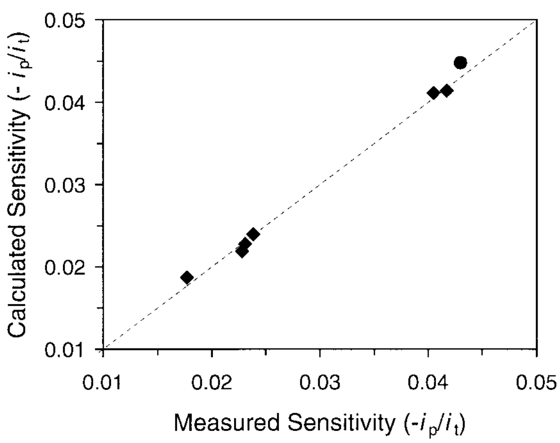

Fig. 4. Comparison of calculated and measured values of the sensitivity for the six CCC constructions reported by Pesel *et al.* [4] (solid diamonds) and the CCC reported by Early and Jones [2] (solid circle).

shield. Both of these effects can be mainly attributed to the way that the toroid current is drawn toward the shield and the pickup coil. The spread of the loops of the pick-up coil (h_p) also enhances the sensitivity.

V. Comparison with Published Measurements

A. Early and Jones

In [2] the authors describe a CCC with a measured sensitivity of −0.043 ± 0.001 compared with a calculated value of −0.054 using the FFT model. By selecting the appropriate parameters (such as making h_t, w_t, and h_p very small) and using the treatment of [2] for the self-inductance of a filament, the strip model calculates the same value as the FFT model. This demonstrates that these two quite different calculation methods are consistent in this limit, and that the FFT approximation in the earlier model was well justified.

For this specific geometry we can now use our detailed model to evaluate the effect of the various contributions that were previously neglected in [2]. In each case below we give the recalculated sensitivity, and in brackets the relative change with respect to the value obtained using all previous refinements. We start with the more precise calculated value of $-0.054\,44$.

1) Including the series inductance of the 66 mm leads connecting the SQUID to the pick-up coil, $L_{\text{leads}} = 0.056\ \mu$H: $-0.052\ 74$ (−3.1%).
2) Allowing for a noninteger number of turns in the pick-up coil (for a fractional turn, L varies approximately as N rather than N^2 [3]): $-0.052\,38$ (−0.7%).
3) Using the GMD to calculate the self-inductance of a strip: $-0.052\ 32$ (−0.1%). This effect is small because a large number of strips (4 096) have been used for the shield.
4) Accounting for the size of the toroid (rather than using a single filament) and including the nonuniform surface current distribution (based on 1 024 strips): $-0.043\,78$ (−16%).
5) Allowing for the spread in the loops of the pick-up coil ($h_p = 6$ mm): $-0.049\ 90$ (+14%).
6) Adding ends to the external shield ($h_s = 49.2$ mm): $-0.049\ 26$ (−1.3%). It is the toroid current distribution for this calculation that is shown in Fig. 2.

The last effect that needs to be included is the presence of the exit tube ($d_e = 7.4$ mm). The toroid current tends to be drawn up, around and down the exit tube, leading to a reduction in the effective coupling between the toroid and the pick-up coil in this region. The close coupling between the shield and exit tube amplifies this effect so that the contribution to the sensitivity from the region where the exit tube is formed is greatly diminished. By comparing the sensitivity for various geometries we estimate this reduction in sensitivity to be 8.9%. This is based on 100% reduction in sensitivity over a sector of length d_e and a 20% reduction over a length of 10 mm on each side of the exit tube (where h_t is greater because of the mechanical support for the windings).

Combining the contributions of all these effects we calculate a value of $-0.044\ 88$ for the sensitivity. Considering that the CCC structure is largely fashioned by manually deforming and soldering lead sheet, the 4% agreement is an excellent result. We note that the six corrections listed above largely cancel (the net change is −9.5%), which explains why the simplistic FFT model gives a reasonable result. Since the first three corrections as well as the exit tube correction can be included in the FFT model, the accuracy of this simpler model may be about 10%.

B. Pesel et al.

The sensitivity for a range of CCC geometries has been measured by Pesel *et al.* [4]. In communication with one of the authors (E. Pesel), we have established the additional parameters required for the strip model (Table II).

Without detailed knowledge of the construction of the CCC's, we have allowed two parameters to vary slightly: L_{leads} and k_e where $k_e d_e$ is the length of the circumference over which the sensitivity is zero because of the exit tube ($d_e = 4.4$ mm). The best fit value for L_{leads} is -0.03 μH, which indicates that this term is negligible compared with the likely variation in the manufacturer's value of $L_{\text{SQ}} = 2.0$ μH. The best fit value for k_e is 3.7, significantly larger than anticipated (the effective value of k_e in section A above is 1.5). However, this term in effect accounts for the error in all of the input parameters (such as the sensitivity of the SQUID) and is therefore probably best thought of as a general correction factor. The calculated sensitivities using these best-fit parameters are given in Table II and displayed in Fig. 4. The agreement is very good with all of the points within 5% of the measured value and four of the six points within 2%. It must be emphasized that the quality of the fit is only weakly dependent on the fitted values of L_{leads} and k_e. For example, using $L_{\text{leads}} = 0.15$ μH and $k_e = 2.7$ only causes a 10% increase in the rms deviation and results in all six points being within 4% of the measured value. Clearly the model works very well for a range of geometries.

C. Hartland

An attempt to establish the feasibility of using a CCC to measure the likely currents that would be produced by SET devices has been reported by Hartland [1]. Although not all of the parameters required for the strip model are given, it is possible to get agreement with the measured sensitivity using reasonable estimates of the unknown values. This then allows us to investigate whether it would have been possible to significantly increase the sensitivity for this CCC.

The calculated sensitivity and the parameters used are shown in Table I where the values that have been estimated are in bold type. To calculate the optimum sensitivity we fix r_s, L_{SQ}, and the area of the toroid, and vary the other parameters. We find that a 30% increase in sensitivity could have been obtained by increasing h_p, and h_s, reducing d_{pt}, and using much heavier wire plus an additional turn in the pick-up coil. Without greatly increasing r_s and reducing the cross sectional area of the windings, it is unlikely any further significant improvement could be made in a practical device, even though this sensitivity is still almost an order of magnitude less than the maximum possible value.

It is instructive to calculate the overall gain of the CCC in terms of the current in the toroid windings. Since the maximum toroid current is $i_t = N_w i_w$ where i_w is the current in each of the N_w windings, the gain is given by $i_p/i_w = N_w(i_p/i_t)$. For Hartland's CCC $N_w = 109\ 999$ or a gain of 9 440, while the CCC in [2] with4 064 turns has a gain of only 175. Although the large CCC is only twice as sensitive ($-0.085\ 8$ compared with -0.043), the factor of 50 improvement in gain arises primarily from increasing N_w. We find that in general it is best to use as many windings as possible (w_t can be up to about 2/3 of r_s), as N_w increases faster than the sensitivity diminishes owing to the increase in the dimensions of the toroid.

VI. Conclusion

A detailed model of a CCC that accounts for the current distribution on the toroid and external shield has been completed. The calculated sensitivity values of the model are in excellent agreement with a wide range of published measurements. This model should be of benefit in the design of CCC's for a variety of applications, and will allow optimal sensitivities to be realized. We find that the scope for significantly increasing the maximum sensitivity in a practical CCC design is quite limited. However, large improvements in the gain of CCC devices can be achieved by increasing the number of windings.

Acknowledgment

The authors are particularly grateful to E. Pesel (PTB) for helpfully providing additional dimensional data for the CCC's described in [4], and to E. M. Frantsuz (VNIIM) for helpful correspondence.

References

[1] A. Hartland, "Development of a cryogenic current comparator for the measurement of small currents," *6th British Electromagnetic Measurements Conf.*, Teddington, U.K., 1993, pp. 18/1–18/4.

[2] M. D. Early and K. Jones, "Optimum sensitivity of an externally shielded cryogenic current comparator," *IEEE Trans. Instrum. Meas.*, vol. 46, pp. 459–462, Apr. 1997.

[3] M. D. Early, M. A. van Dam, and K. Jones, *A Detailed Model of the Sensitivity of a Cryogenic Current Comparator*, unpublished.

[4] E. Pesel, P. Warnecke, and Z. Zhang, "Binary 13-bit cryogenic current comparator with nonsuperconducting windings," in *Proc. 2nd Int. Symp. Electromagnetic Metrology*, Beijing, China, 1993, pp. 49–53.

[5] F. W. Grover, *Inductance Calculations*. New York: Dover, 1962.

[6] G. T. Symm, "Design of a cryogenic current comparator," in *Proc. Boundary Elements XIV Vol. 1: Field Problems and Applications*, C. A. Brebbia, J. Dominguez, and F. Paris, Eds., 1992, pp. 519–526.

Murray D. Early was born April 20, 1959, in Wanganui, New Zealand. He received the B.Sc.(Hons.) and the Ph.D. degrees in physics from Victoria University, Wellington, New Zealand, in 1981 and 1988, respectively.

He was a Junior Lecturer in the Victoria University Physics Department, from 1985 until 1987. From 1988 to 1990, he was with the Aerodynamics Section, Works Central Laboratories, Lower Hutt, New Zealand, performing investigations into aerodynamic effects on tall buildings. He joined the Department of Scientific and Industrial Research, Lower Hutt, in 1990, taking responsibility for ac electrical standards. He is presently setting up a primary ac voltage standard at the Measurement Standards Laboratory of New Zealand (MSL) and continuing with his interests in modeling the behavior of CCC's.

Marcos A. van Dam was born in Buenos Aires, Argentina on July 11, 1976. He is pursuing the B.Sc. degree in physics and the B.E.(Hons.) degree in electrical and electronic engineering at the University of Canterbury, Christchurch, New Zealand. His research interests are very broad, ranging from electromagnetic modeling through to signal processing and error-control coding in communication systems. He intends to pursue the Ph.D. degree in adaptive optics in 1999 at the University of Canterbury.